Advanced action in classical electrodynamics

Advanced action in classical electrodynamics

A D Boozer

Department of Physics, California Institute of Technology, Pasadena, CA 91125, USA

E-mail: boozer@caltech.edu

Abstract

The time evolution of a charged point particle is governed by a second-order integro-differential equation that exhibits advanced effects, in which the particle responds to an external force before the force is applied. In this paper, we give a simple argument that clarifies the origin and physical meaning of these advanced effects, and we compare ordinary electrodynamics with a toy model of electrodynamics in which advanced effects do not occur.

PACS numbers: 03.50.−z, 03.50.De, 03.50.Kk, 11.10.−z, 11.10.Kk

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Charged point particles in classical electrodynamics obey an equation of motion, known as the Lorentz–Dirac equation, that is third order in proper time, and therefore presents various difficulties [1–4]. In particular, the Lorentz-Dirac equation admits unphysical runaway solutions, in which the particle accelerates without bound even in the absence of an externally imposed force. The runaway solutions can be eliminated by replacing the Lorentz–Dirac equation with a second-order integro-differential equation, but this equation has problems of its own: it exhibits advanced effects, in which the particle responds to an external force before the force has been applied.

In this paper, we give a simple argument that clarifies the origin and physical meaning of these advanced effects, and we compare ordinary electrodynamics with a toy model of electrodynamics in which advanced effects do not occur. The toy model is closely analogous to ordinary electrodynamics; for example, there are fields that correspond to the electric and magnetic fields, and these fields mediate an interaction between charged particles and support freely propagating radiation. The toy model, however, does not exhibit the conceptual problems that plague ordinary electrodynamics; for example, the self-energy of a point particle does not diverge, and there are no advanced effects. Thus, the toy model shows that it is possible to construct a mathematically consistent theory of coupled particles and fields that

is not subject to these pathologies. By performing a detailed comparison between the toy model and ordinary electrodynamics, we can gain insight into the conceptual foundations of electrodynamics and into the meaning of the advanced effects.

This paper is organized as follows. In section 2, we briefly review the Lorentz–Dirac equation of motion, the corresponding integro-differential equation of motion and the causality problems associated with it. In section 3, we consider a spatially extended charged particle and give a simple physical argument that explains the origin of the advanced effects. In section 4, we introduce the toy model of electrodynamics. In section 5, we consider a spatially extended particle in the toy model and derive analogs to the various equations of motion for electrodynamics.

The following notation is used in this paper. The function $\epsilon(x)$ is the sign function, defined such that $\epsilon(x) = 1$ if $x > 0$, $\epsilon(x) = 0$ if $x = 0$, and $\epsilon(x) = -1$ if $x < 0$. The function $\theta(x)$ is the step function, defined such that $\theta(x) = 1$ if $x > 0$, $\theta(x) = 1/2$ if $x = 0$, and $\theta(x) = 0$ if $x < 0$. The metric tensor $\eta_{\mu\nu}$ is defined such that $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. The 3-vector component of a 4-vector is denoted by bold-face type; for example, $x^\mu = (x^0, \mathbf{x})$.

2. The Lorentz–Dirac equation

Consider a point particle coupled to the electromagnetic field. We will let m and e denote the mass and charge of the particle and let $z^\mu(\tau)$ denote the position of the particle at proper time τ. Also, we will define $v^\mu = dz^\mu/d\tau$ to be the velocity of the particle and $a^\mu = dv^\mu/d\tau$ to be its acceleration. The equation of motion for the particle is

$$ma^\mu = K_f^\mu + K_{\text{ext}}^\mu, \tag{1}$$

where K_f^μ is the force exerted on the particle by the electromagnetic field and K_{ext}^μ is an arbitrary externally imposed force. For simplicity, we will assume that K_{ext}^μ depends only on the proper time τ, and not on the position or velocity of the particle. The force K_f^μ is given by

$$K_f^\mu(\tau) = e F^{\mu\nu}(z(\tau)) v_\nu(\tau), \tag{2}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field-strength tensor and A^μ is the vector potential. In the Lorentz gauge ($\partial_\mu A^\mu = 0$) the vector potential satisfies the field equation

$$\Box A^\mu = 4\pi J^\mu, \tag{3}$$

where $J^\mu(x)$, the current density, is given by

$$J^\mu(x) = e \int v^\mu(\tau) \delta^{(4)}(x - z(\tau)) \, d\tau. \tag{4}$$

Equations (1)–(4) give a complete description of the coupled particle-field system.

We can decompose each solution to the field equation (3) into the sum of an inhomogeneous solution, which describes the potential generated by the particle, and a homogeneous solution, which describes freely propagating radiation. It is useful to perform this decomposition in two different ways:

$$A^\mu = A_r^\mu + A_{\text{in}}^\mu = A_a^\mu + A_{\text{out}}^\mu, \tag{5}$$

where A_r^μ and A_a^μ are the retarded and advanced potentials generated by the particle, and A_{in}^μ and A_{out}^μ describe incoming and outgoing radiation. The retarded and advanced potentials are given by

$$A_r^\mu(x) = \int D_r(x - x') J^\mu(x') \, d^4x', \tag{6}$$

$$A_a^\mu(x) = \int D_a(x - x') J^\mu(x') \, \mathrm{d}^4 x', \tag{7}$$

where $D_r(x)$ and $D_a(x)$ are the retarded and advanced Green functions for the inhomogeneous wave equation:

$$D_r(x) = 2\theta(x^0)\delta(x \cdot x) = |\mathbf{x}|^{-1}\delta(x^0 - |\mathbf{x}|), \tag{8}$$

$$D_a(x) = 2\theta(-x^0)\delta(x \cdot x) = |\mathbf{x}|^{-1}\delta(x^0 + |\mathbf{x}|). \tag{9}$$

Using the decompositions of the vector potential given in equation (5), we can express the field-strength tensor and the electromagnetic force as

$$F^{\mu\nu} = F_r^{\mu\nu} + F_{\text{in}}^{\mu\nu} = F_a^{\mu\nu} + F_{\text{out}}^{\mu\nu}, \tag{10}$$

$$K_f^\mu = K_r^\mu + K_{\text{in}}^\mu = K_a^\mu + K_{\text{out}}^\mu, \tag{11}$$

where, for example,

$$F_r^{\mu\nu} = \partial^\mu A_r^\nu - \partial^\nu A_r^\mu, \qquad K_r^\mu(\tau) = e F_r^{\mu\nu}(z(\tau)) v_\nu(\tau). \tag{12}$$

Physically, K_{in}^μ describes the force exerted on the particle by incoming radiation, and K_r^μ describes the self-force exerted on the particle by its own retarded field. In what follows, we will assume that there is no incoming radiation, so $K_{\text{in}}^\mu = 0$ and $K_f^\mu = K_r^\mu$.

We can obtain an explicit expression for the self-force by combining equations (4), (6) and (12). To perform the calculation it is useful to express the self-force in the form $K_r^\mu = K_+^\mu + K_-^\mu$, where $K_\pm^\mu \equiv (1/2)\left(K_r^\mu \pm K_a^\mu\right)$. As was first shown by Dirac [4] (see also [3], pages 187–9), the component K_-^μ is well defined and is given by

$$K_-^\mu = m\tau_0(\dot{a}^\mu + (a \cdot a)v^\mu), \tag{13}$$

where $\dot{a}^\mu \equiv \mathrm{d}a^\mu/\mathrm{d}\tau$ and $\tau_0 \equiv (2/3)(e^2/m)$. The component K_+^μ, however, is infinite. The reason for this can be traced to the fact that the retarded fields of a point particle diverge as we approach the particle; for a spatially extended particle, one can show that K_+^μ is finite and is given by

$$K_+^\mu = -m_S a^\mu + \cdots, \tag{14}$$

where m_S is the self-energy of the particle and the dots indicate additional terms that vanish in the point particle limit. We derive an expression for the self-energy of a spherically symmetric particle in appendix A, but for now we simply note that the self-energy scales like $1/\sigma$, where σ is the particle size, and therefore diverges in the point particle limit. Thus, in order to obtain a finite expression for the self-force, let us assume that the particle is spatially extended, but small enough that K_-^μ is well approximated by equation (13) and K_+^μ is well approximated by the first term of equation (14):

$$K_r^\mu = K_-^\mu + K_+^\mu = m\tau_0(\dot{a}^\mu + (a \cdot a)v^\mu) - m_S a^\mu. \tag{15}$$

Substituting this result into equation (1), we find that the equation of motion for the particle is

$$ma^\mu = m\tau_0(\dot{a}^\mu + (a \cdot a)v^\mu) - m_S a^\mu + K_{\text{ext}}^\mu. \tag{16}$$

Here m is the bare mass of the particle; that is, the mass that the particle would have if its charge were set to zero. Let us define a renormalized mass $m_R \equiv m + m_S$ and a renormalized time constant $\tau_R \equiv (m/m_R)\tau_0 = (2/3)(e^2/m_R)$. We can then express the equation of motion as

$$a^\mu - \tau_R \dot{a}^\mu = \tau_R(a \cdot a)v^\mu + (1/m_R)K_{\text{ext}}^\mu. \tag{17}$$

This is the Lorentz–Dirac equation. We derived this equation by considering a small spatially extended particle, but it is well defined in the point particle limit, provided we hold m_R constant and allow m to diverge in order to compensate for the divergence of m_S.

It is instructive to consider the low-velocity limit of the Lorentz–Dirac equation:

$$\mathbf{a} - \tau_R \dot{\mathbf{a}} = (1/m_R)\mathbf{K}_{\text{ext}}, \tag{18}$$

where $\dot{\mathbf{a}} \equiv \mathrm{d}\mathbf{a}/\mathrm{d}t$. An ordinary equation of motion is second order in time, and thus requires initial conditions $\mathbf{z}_0$ and $\mathbf{v}_0$ for the position and velocity, but since equation (18) is third order in time we need an additional initial condition $\mathbf{a}_0$ for the acceleration. If $\mathbf{a}_0$ is not chosen properly we obtain unphysical runaway solutions. We can see this for the case of a free particle ($\mathbf{K}_{\text{ext}} = 0$), for which the solution to equation (18) is

$$\mathbf{a}(t) = \mathbf{a}_0 \, \mathrm{e}^{t/\tau_R}. \tag{19}$$

Thus, unless $\mathbf{a}_0 = 0$ we obtain a runaway solution in which the acceleration increases exponentially in time. We can eliminate the runaway solutions by expressing the solution to equation (18) in terms of a Green function $G(t)$:

$$\mathbf{a}(t) = (1/m_R)\int G(t - t')\mathbf{K}_{\text{ext}}(t')\,\mathrm{d}t', \tag{20}$$

where $G(t)$ is defined such that $G(t) \to 0$ for $t \to \pm\infty$ and

$$G(t) - \tau_R \dot{G}(t) = \delta(t). \tag{21}$$

Let $\tilde{G}(\omega)$ denote the Fourier transform of $G(t)$. From equation (21), it follows that

$$\tilde{G}(\omega) = \int G(t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = (1 + \mathrm{i}\tau_R\omega)^{-1}, \tag{22}$$

$$G(t) = (2\pi)^{-1}\int \tilde{G}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = (1/\tau_R)\theta(-t)\,\mathrm{e}^{t/\tau_R}. \tag{23}$$

We can view equation (20) as a new equation of motion for the particle. By its construction it is a solution to equation (18), so integrating equation (20) subject to the initial conditions $\{\mathbf{z}_0, \mathbf{v}_0\}$ is equivalent to integrating equation (18) subject to the initial conditions $\{\mathbf{z}_0, \mathbf{v}_0, \mathbf{a}_0\}$, where $\mathbf{a}_0$ is given by

$$\mathbf{a}_0 = (m_R\tau_R)^{-1}\int_0^\infty \mathrm{e}^{-t'/\tau_R}\mathbf{K}_{\text{ext}}(t')\,\mathrm{d}t'. \tag{24}$$

In effect, equation (20) selects the initial condition $\mathbf{a}_0$ so as to eliminate the runaway solutions. We can see this for the case of a free particle: when $\mathbf{K}_{\text{ext}} = 0$, equation (24) sets $\mathbf{a}_0 = 0$. Note that since $G(t) > 0$ for $t < 0$, equation (20) exhibits advanced effects, in which the acceleration of the particle at time t depends on the value of the external force at times $t' > t$.

For simplicity, we have shown how the runaway solutions can be eliminated in the low-velocity limit, but the same method can be applied to the full relativistic Lorentz-Dirac equation (see section 6.6 of [1], section 5.6 of [3], and [5]). The resulting equation of motion is

$$a^\mu(\tau) = \int_\tau^\infty \mathrm{e}^{-(\tau'-\tau)/\tau_R}\left((a(\tau') \cdot a(\tau'))v^\mu(\tau') + (m_R\tau_R)^{-1}K_{\text{ext}}^\mu(\tau')\right)\mathrm{d}\tau'. \tag{25}$$

Note that $a^\mu(\tau)$ depends on the value of $K_{\text{ext}}^\mu(\tau')$ at proper times $\tau' > \tau$, so equation (25) also exhibits advanced effects.

3. Extended particles in electrodynamics

We can gain some insight into the origin of the advanced effects by generalizing equation (25) to the case of a spatially extended particle of arbitrary size. For simplicity, let us consider an extended particle that is spherically symmetric. The current density is then given by

$$J^\mu(x) = e \int (1 - a(\tau) \cdot (x - z(\tau))) f(-(x - z(\tau))^2) v^\mu(\tau) \delta(v(\tau) \cdot (x - z(\tau))) \, \mathrm{d}\tau, \qquad (26)$$

where e is the total charge of the particle and $f(r^2)$ describes the radial charge distribution (this expression for the current density is derived in section 7.4 of [1]). It is convenient to express the current density in the form

$$J^\mu(x) = \int \bar{J}^\mu(x, \tau) \, \mathrm{d}\tau, \qquad (27)$$

where

$$\bar{J}^\mu(x, \tau) \equiv e(1 - a(\tau) \cdot (x - z(\tau))) f(-(x - z(\tau))^2) v^\mu(\tau) \delta(v(\tau) \cdot (x - z(\tau))). \qquad (28)$$

Note that $\bar{J}^\mu(x, \tau)$ vanishes unless x^μ lies in the plane of simultaneity for proper time τ. We can describe the coupling of the particle to the electromagnetic field using the action

$$S_i = -\int A^\mu(x) J_\mu(x) \, \mathrm{d}^4 x = \int L_i \, \mathrm{d}\tau, \qquad (29)$$

where

$$L_i = -\int A^\mu(x) \bar{J}_\mu(x, \tau) \, \mathrm{d}^4 x \qquad (30)$$

is the corresponding Lagrangian. From the Euler–Lagrange equations, it follows that the electromagnetic force on the extended particle is

$$K_f^\mu(\tau) = \int F^{\mu\nu}(x) \bar{J}_\nu(x, \tau) \, \mathrm{d}^4 x. \qquad (31)$$

As before, we can separate the electromagnetic force into a component K_r^μ that describes the self-force and a component K_{in}^μ that describes the force exerted on the particle by incoming radiation. Using equation (6) to solve for $A_r^\mu(x)$ in terms of the current density, we find that the self-force is given by

$$K_r^\mu(\tau) = \int F_r^{\mu\nu}(x) \bar{J}_\nu(x, \tau) \, \mathrm{d}^4 x = \int \bar{K}_r^\mu(\tau, \tau') \, \mathrm{d}\tau', \qquad (32)$$

where we have defined

$$\bar{K}_r^\mu(\tau, \tau') \equiv \iint \bar{J}^\alpha(x, \tau) \Delta^\mu{}_{\alpha\beta}(x, x') \bar{J}^\beta(x', \tau') \, \mathrm{d}^4 x \, \mathrm{d}^4 x' \qquad (33)$$

and

$$\Delta^\mu{}_{\alpha\beta}(x, x') \equiv \left(\eta_{\alpha\beta} \partial^\mu - \eta^\mu{}_\beta \partial_\alpha \right) D_r(x - x'). \qquad (34)$$

If we assume that there is no incoming radiation, the equation of motion for the extended particle is

$$ma^\mu = K_r^\mu + K_{\mathrm{ext}}^\mu, \qquad (35)$$

where K_r^μ is given by equation (32) and K_{ext}^μ describes an arbitrary externally imposed force. This is a second-order integro-differential equation; it is exact, and holds for extended particles of arbitrary size. The equation of motion (25) that we derived in the previous section should approximate this equation of motion in the limit of a small particle (the parameters
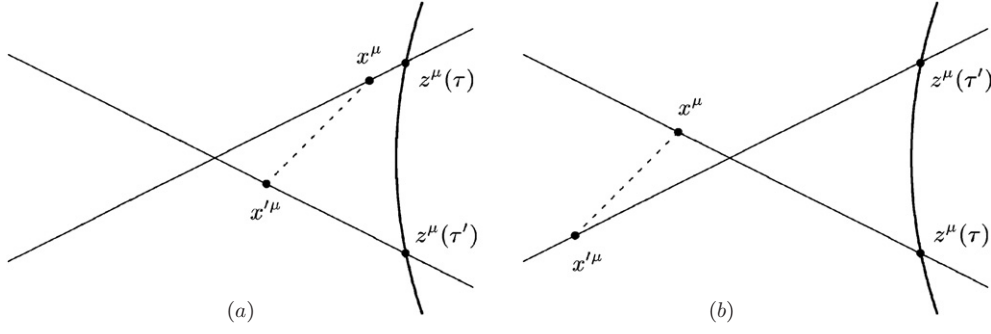
Figure 1. Spacetime diagrams, which illustrate pairs of events x^μ and x'^μ that give nonzero contributions to $\bar{K}_r^\mu(\tau, \tau')$. The thick solid curves indicate the particle trajectory, the thin solid lines indicate the planes of simultaneity for proper times τ and τ', the dashed lines indicate the light-like interval between events x^μ and x'^μ. (a) Conditions (1)–(3) are satisfied for $\tau > \tau'$, corresponding to retarded effects. (b) Conditions (1)–(3) are satisfied for $\tau < \tau'$, corresponding to advanced effects.

$m_R = m + m_S$ and $\tau_R = (2/3)(e^2/m_R)$ that appear in equation (25) are set by the self-energy m_S, which can be obtained from $f(r^2)$ using equation (A.10)).

In general, the self-force acting on the particle at proper time τ depends on the state of the particle at proper times both earlier and later than τ. We can understand this by examining equation (33) for $\bar{K}_r^\mu(\tau, \tau')$. From the three factors in the integrand, it follows that $\bar{K}_r^\mu(\tau, \tau')$ vanishes unless three conditions are met: (1) there is an event x^μ within the world cylinder of the particle that lies on the plane of simultaneity for proper time τ, (2) there is an event x'^μ within the world cylinder of the particle that lies on the plane of simultaneity for proper time τ', (3) events x^μ and x'^μ are light-like separated, with x'^μ earlier than x^μ. There are retarded effects when $\bar{K}_r^\mu(\tau, \tau')$ is nonzero for $\tau > \tau'$, and advanced effects when it is nonzero for $\tau < \tau'$ (see figure 1).

We can give a concrete example that illustrates these conditions by considering a uniformly accelerated particle in (1+1) dimensions. The position, velocity and acceleration of the particle are given by

$$z^\mu(\tau) = a^{-1}e_1^\mu(\tau), \qquad v^\mu(\tau) = e_0^\mu(\tau), \qquad a^\mu(\tau) = ae_1^\mu(\tau), \qquad (36)$$

where

$$e_0^\mu(\tau) \equiv (\cosh a\tau, \sinh a\tau), \qquad e_1^\mu(\tau) \equiv (\sinh a\tau, \cosh a\tau). \qquad (37)$$

Note that $e_0 \cdot e_0 = -e_1 \cdot e_1 = 1$ and $e_0 \cdot e_1 = 0$. It is convenient to define a new coordinate system (λ, u) by

$$x^\mu(\lambda, u) = z^\mu(\lambda) + ue_1^\mu(\lambda) = (a^{-1} + u)e_1^\mu(\lambda). \qquad (38)$$

The trajectory of the particle is then given by $z^\mu(\tau) = x^\mu(\tau, 0)$, and the plane of simultaneity for proper time τ is given by $x^\mu(\tau, u)$; note that the planes of simultaneity for all values of τ pass through the origin at $u = -1/a$. Equation (28) for $\bar{J}^\mu(x, \tau)$ generalizes naturally to $(1 + 1)$ dimensions; from the above expressions, it follows that

$$\bar{J}^\mu(x, \tau) = e(1 + au)f(u^2)e_0^\mu(\tau)\delta(\lambda - \tau). \qquad (39)$$

We will assume that $f(u^2) > 0$ for $u < R$ and $f(u^2) = 0$ for $u > R$, so the trajectories of the left and right edges of the particle are given by $z_-^\mu(\lambda) \equiv x^\mu(\lambda, -R)$ and $z_+^\mu(\lambda) \equiv x^\mu(\lambda, +R)$.
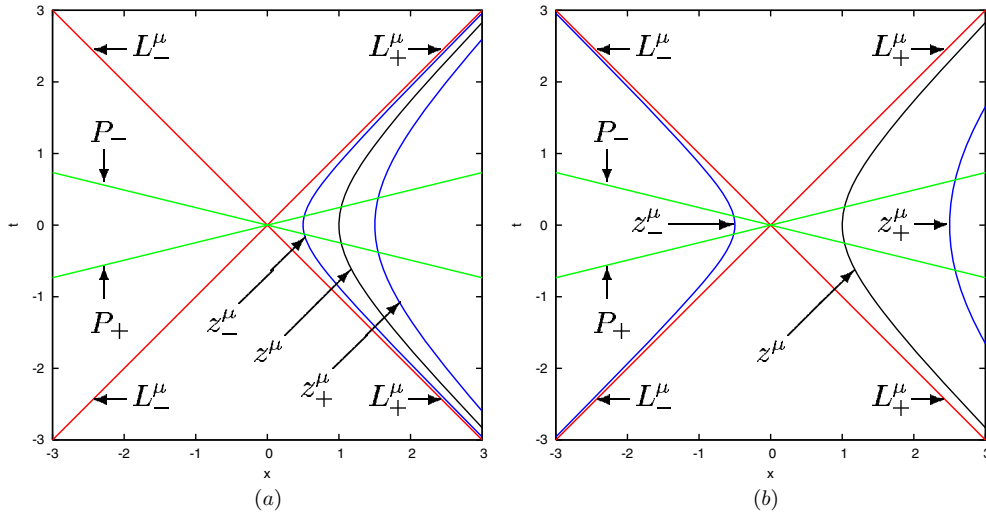
Figure 2. Spacetime diagrams for a uniformly accelerated particle of radius R. Black curve: particle trajectory z^μ. Blue curves: trajectories z^μ_- and z^μ_+ of the left and right edges of the particle. Red lines: trajectories L^μ_+ and L^μ_-. Green lines: planes of simultaneity $P_\pm$ for $\tau = \pm 1/4$. (a) $aR = 1/2$. (b) $aR = 3/2$.

If $aR < 1$, then the charge density is positive in the region between the trajectories z^μ_- and z^μ_+, and zero everywhere else (see figure 2(a)). Thus, conditions (1)–(3) are met only if $\tau > \tau'$. If $aR > 1$, then the situation is more complicated. Let us define trajectories $L^\mu_\pm(\lambda) = (\lambda, \pm|\lambda|)$; these trajectories define the left and right edges of the forward and backward light cones for the origin. Also, let us define R_+ to be the region between L^μ_+ and z^μ_+, and R_- to be the region between z^μ_- and L^μ_-. The charge density is positive in the region R_+, negative in the region R_-, and zero everywhere else (see figure 2(b)). Thus, conditions (1)-(3) are always met: the events x^μ and x'^μ lie in region R_+ if $\tau > \tau'$, and in region R_- if $\tau < \tau'$.

4. The toy model of electrodynamics

We can gain further insight into the advanced effects by comparing ordinary electrodynamics with a toy model of electrodynamics in which advanced effects do not occur. A complete description of the model is given in [6], but all the results that we will need are summarized here.

The toy model that we will be considering describes a spatially extended particle in $(1 + 1)$ dimensions that obeys Newtonian dynamics and is coupled to a pair of fields $E(t, x)$ and $B(t, x)$, which correspond to the electric and magnetic fields of ordinary electrodynamics. The equations of motion for these fields are

$$\partial_t E(t, x) = \partial_x B(t, x), \tag{40}$$

$$\partial_t B(t, x) = \partial_x E(t, x) - 2\rho(t, x), \tag{41}$$

where $\rho(t, x)$ is the charge density. We will assume that the charge density has the form

$$\rho(t, x) = gf(x - z(t)), \tag{42}$$

where $z(t)$ is the position of the particle at time t, g is its charge and $f(x)$ describes the charge distribution. The equation of motion for the particle is

$$m\ddot{z} = F_f + F_{\text{ext}}, \tag{43}$$

where m is the particle mass,

$$F_f(t) = -2 \int \rho(t, x) E(t, x) \, \mathrm{d}x \tag{44}$$

is the force that the E-field exerts on the particle and F_{ext} describes an arbitrary externally imposed force. Equations (40)–(44) give a complete description of the coupled particle-field system. Equations (40) and (41) can be thought of as the analogs to Maxwell's equations and equation (44) can be thought of as the analog to the Lorentz force law.

By analogy with electrodynamics, we can express the E and B fields in the form

$$E(t, x) = E_r(t, x) + E_{\text{in}}(t, x), \qquad B(t, x) = B_r(t, x) + B_{\text{in}}(t, x), \tag{45}$$

where $E_r(t, x)$ and $B_r(t, x)$ are the retarded fields generated by the particle, and $E_{in}(t, x)$ and $B_{in}(t, x)$ describe incoming radiation (see [7]). The retarded fields are given by

$$E_r(t, x) = \partial_x \phi_r(t, x), \qquad B_r(t, x) = \partial_t \phi_r(t, x), \tag{46}$$

where

$$\phi_r(t, x) \equiv - \iint D_r(t - t', x - x') \rho(t', x') \, \mathrm{d}t' \, \mathrm{d}x' \tag{47}$$

and $D_r(t, x) = \theta(t - |x|)$ is the retarded Green function for the inhomogeneous wave equation in $(1 + 1)$ dimensions. Using the decompositions given in equation (45), we can express the force exerted by the field as $F_f = F_{\text{in}} + F_r$, where

$$F_{\text{in}}(t) = -2 \int \rho(t, x) E_{in}(t, x) \, \mathrm{d}x, \tag{48}$$

$$F_r(t) = -2 \int \rho(t, x) E_r(t, x) \, \mathrm{d}x. \tag{49}$$

Physically, F_{in} describes the force exerted on the particle by incoming radiation, and F_r describes the self-force exerted on the particle by its own retarded field. In what follows we will assume that there is no incoming radiation, so $F_{\text{in}} = 0$ and $F_f = F_r$.

Using equations (46), (47) and (49), we can evaluate the self-force explicitly for the case of a point particle, for which $f(x) = \delta(x)$:

$$F_r = -m \gamma \dot{z} (1 - \dot{z}^2)^{-1}, \tag{50}$$

where $\gamma \equiv 2g^2/m$ is a damping constant. From equations (43) and (50), we find that the equation of motion is

$$\ddot{z} + \gamma \dot{z} (1 - \dot{z}^2)^{-1} = (1/m) F_{\text{ext}}. \tag{51}$$

This is the toy model analog to the Lorentz–Dirac equation (17) for electrodynamics. If we compare equation (51) with the Lorentz–Dirac equation, we note three important differences. First, the Lorentz–Dirac equation is of third order, but equation (51) is only of second order, and thus it does not admit runaway solutions and does not need to be replaced with an integro-differential equation. In the toy model the acceleration of a point particle at time t only depends on the value of F_{ext} at time t, and there are no advanced effects. Second, for the Lorentz–Dirac equation there is a radiation damping term proportional to the time derivative of the acceleration, while for equation (51) the damping term is a function of the particle velocity.
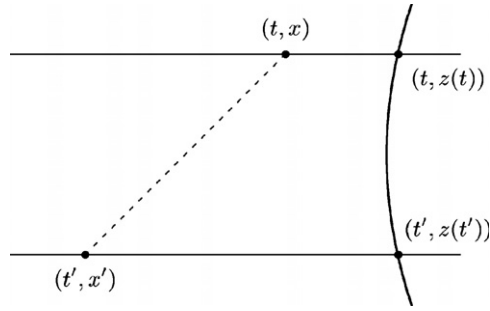
Figure 3. Spacetime diagram, which illustrates a pair of events (t, x) and (t', x') that give a nonzero contribution to $\bar{F}_r(t, t')$. The thick solid curve indicates the particle trajectory; the thin solid lines indicate the planes of simultaneity for times t and t'; the dashed line indicates the light-like interval between events (t', x') and (t, x).

For electrodynamics, a velocity-dependent damping term is ruled out by Lorentz invariance, but the toy model is neither Lorentz nor Galilean invariant; there is a preferred reference frame in which the equations of motion for the model are valid, and a particle moving with respect to this preferred frame feels a velocity-dependent drag force. Third, the mass that appears in the Lorentz–Dirac equation is the renormalized mass, the sum of the bare mass and the self-energy, while the mass that appears in equation (51) is just the bare mass; there is no self-energy contribution. We will explain the reason for this in section 5.1.

5. Extended particles in the toy model

Let us now consider a spatially extended particle in the toy model. From equations (46), (47) and (49), it follows that the self-force for an extended particle is given by

$$F_r(t) = \int \bar{F}_r(t, t') \, \mathrm{d}t', \tag{52}$$

where we have defined

$$\bar{F}_r(t, t') = 2 \iint \rho(t, x) \partial_x D_r(t - t', x - x') \rho(t', x') \, \mathrm{d}x \, \mathrm{d}x'. \tag{53}$$

Equations (52) and (53) are the toy model analogs of equations (32) and (33) for ordinary electrodynamics. Note, however, that whereas $\bar{K}_r(\tau, \tau')$ can be nonzero for either $\tau > \tau'$ or $\tau < \tau'$, the analogous quantity $\bar{F}_r(t, t')$ is only nonzero if $t > t'$. Physically, this is due to the fact that the particle obeys Newtonian dynamics, so the planes of simultaneity do not tilt (see figure 3, and compare with figure 1 for electrodynamics). In what follows, we will assume that the charge distribution of the extended particle is

$$f(x) = (2\pi\sigma^2)^{-1/2} \mathrm{e}^{-x^2/2\sigma^2}, \tag{54}$$

where σ describes the particle size. If we substitute equation (54) into equation (53), we find that

$$\bar{F}_r(t, t') = (m\gamma/2\sqrt{\pi}\sigma)\theta(t - t')(\mathrm{e}^{-(z(t)-z(t')+t-t')^2/4\sigma^2} - \mathrm{e}^{-(z(t)-z(t')+t'-t)^2/4\sigma^2}). \tag{55}$$

The equation of motion for an extended particle is

$$m\ddot{z} = F_r + F_{\text{ext}}, \tag{56}$$

where F_r is given by equation (52) and F_{ext} describes an externally imposed force. This is the toy model analog to equation (35), the equation of motion for an extended particle in electrodynamics. Like equation (35), it is a second-order integro-differential equation that is exact and holds for extended particles of arbitrary size. Given the velocity v_0 at time t_0, together with the particle trajectory $z(t)$ at all times $t \leqslant t_0$, equation (56) can be integrated to obtain the particle trajectory at all times.

5.1. Approximate equation of motion

We can further develop the analogy between the toy model and electrodynamics by performing a series expansion of the self-force for an extended particle. If we substitute equation (53) into equation (52) and perform the integral over t', we find that

$$F_r(t) = -2 \iint \epsilon(x - x')\rho(t, x)\rho(t - |x - x'|, x') \, dx \, dx'. \tag{57}$$

Let us expand $\rho(t - |x - x'|, x')$ in $|x - x'|$:

$$\rho(t - |x - x'|, x') = \sum_{n=0}^{\infty} (1/n!)(-1)^n |x - x'|^n \partial_t^n \rho(t, x'). \tag{58}$$

If we assume that the particle is moving slowly, then we can neglect terms that are nonlinear in $z^{(n)}(t) \equiv d^n z(t)/dt^n$:

$$\rho(t - |x - x'|, x') = \rho(t, x') - \sum_{n=1}^{\infty} (1/n!)(-1)^n |x - x'|^n z^{(n)}(t) \partial_{x'} \rho(t, x'). \tag{59}$$

Substituting equation (59) into equation (57), we find that

$$F_r(t) = -m\gamma \sum_{n=0}^{\infty} (-1)^n c_n \sigma^n v^{(n)}(t), \tag{60}$$

where $v^{(n)} \equiv d^n v(t)/dt^n$, $v = \dot{z}$ is the particle velocity, and

$$c_n \equiv (1/n!\sigma^n) \iint |x - x'|^n f(x) f(x') \, dx \, dx' = \frac{2^n}{\sqrt{\pi}} \frac{\Gamma(n/2 + 1/2)}{\Gamma(n + 1)} \tag{61}$$

are dimensionless coefficients. The first few terms of equation (60) are

$$F_r = -m\gamma v - m_S \dot{v} + m\tau_0 \ddot{v} + \cdots, \tag{62}$$

where $m_S \equiv -(4/\sqrt{\pi}) g^2 \sigma$ and $\tau_0 \equiv -2g^2\sigma^2/m$. The first term is just the low-velocity limit of the self-force for a point particle, and the second term describes the self-energy m_S. Since the self-energy is proportional to the particle size, it vanishes in the point particle limit, which explains why it is the bare mass, rather than the renormalized mass, that appears in the point particle equation of motion (51).

It is instructive to compare equation (62) with the the low-velocity limit of the electrodynamic self-force given in equation (15):

$$\mathbf{K}_r = -m_S \mathbf{a} + m\tau_0 \dot{\mathbf{a}} + \cdots, \tag{63}$$

where the dots indicate terms that vanish in the point particle limit. Note that for electrodynamics the term proportional to the velocity is not present (as we discussed before, such a term is ruled out by Lorentz invariance). Also, the self-energy m_S and time constant τ_0 are both negative for the toy model, but positive for electrodynamics. These quantities also scale differently in the two theories: for the toy model, $m_S \sim \sigma$ and $\tau_0 \sim \sigma^2$, whereas

for electrodynamics $m_S \sim 1/\sigma$ and τ_0 is independent of the particle size. Thus, in the point particle limit,

$$F_r \rightarrow -m\gamma v, \qquad \mathbf{K}_r \rightarrow -m_S \mathbf{a} + m\tau_0 \dot{\mathbf{a}}. \tag{64}$$

For a small enough particle, we can obtain a good approximation to equation (60) by truncating the series at some finite order N:

$$F_r = -m\gamma \sum_{n=0}^{N-1} (-1)^n c_n \sigma^n v^{(n)}. \tag{65}$$

Thus, we obtain an approximate equation of motion for the particle,

$$v^{(1)} + \gamma \sum_{n=0}^{N-1} (-1)^n c_n \sigma^n v^{(n)} = (1/m) F_{\text{ext}}. \tag{66}$$

This should be a good approximation to the exact equation of motion (56) provided the particle is small and slowly moving. Since this equation is of order N, we need initial conditions $z^{(n)}(0)$ for $n = 0, 1, \ldots, N-1$.

We can find the solutions to equation (66) for the special case $F_{\text{ext}} = 0$:

$$v(t) = \sum_{k=1}^{N-1} A_k \, e^{-\beta_k t/\sigma}, \tag{67}$$

where the constants $A_1, \ldots, A_{N-1}$ are set by the initial conditions, the constants $\beta_1, \ldots, \beta_{N-1}$ are the $N-1$ roots of the polynomial

$$p(\beta) = \beta - \eta \sum_{n=0}^{N-1} c_n \beta^n, \tag{68}$$

and $\eta \equiv \gamma\sigma$ is a dimensionless measure of the particle size. Note that some of the roots may be complex, and if so then the initial conditions must be chosen such that $v(t)$ is real (the fact that there are complex roots suggests that there are oscillatory solutions to the equation of motion; examples of such solutions are given in appendix B). Also, note that if $\text{Re } \beta_k < 0$ and $A_k \neq 0$, then we obtain a runaway solution in which the velocity increases exponentially in time. We can solve for the roots in the limit of small particle size ($\eta \ll 1$):

$$\beta_k = e^{2\pi i(k-1)/(N-2)} (\eta c_{N-1})^{-1/(N-2)} \qquad \text{for} \quad k = 1, \ldots, N-2 \tag{69}$$

$$\beta_{N-1} = \eta. \tag{70}$$

Note that for $N \geqslant 4$ there is at least one root that yields a runaway solution. For the case $N = 3$ we can calculate the roots exactly and write the solutions explicitly; this is done in appendix C.

5.2. The integro-differential approximate equation of motion

As for electrodynamics, we can eliminate the runaway solutions by replacing the approximate equation of motion (66) with an integro-differential equation. Let us define a Green function $G(t)$ by

$$G^{(1)}(t) + \gamma \sum_{n=0}^{N-1} (-1)^n c_n \sigma^n G^{(n)}(t) = \delta(t). \tag{71}$$

We will assume that $N \geqslant 3$, so that the polynomial $p(\beta)$ defined in equation (68) can be factorized as follows:

$$p(\beta) = \beta - \eta \sum_{n=0}^{N-1} c_n \beta^n = -\eta c_{N-1} \prod_{n=1}^{N-1} (\beta - \beta_n). \tag{72}$$

Using this result, it is straightforward to show that the Fourier transform $\tilde{G}(\omega)$ of $G(t)$ is

$$\tilde{G}(\omega) = \int G(t)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}t = -\frac{\sigma}{p(\mathrm{i}\omega\sigma)} = (c_{N-1}\gamma)^{-1} \prod_{k=1}^{N-1} (\mathrm{i}\omega\sigma - \beta_k)^{-1}. \tag{73}$$

Thus, $G(t)$ is given by

$$G(t) = (2\pi)^{-1} \int \tilde{G}(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}\omega = \sum_{k=1}^{N-1} B_k\, \mathrm{e}^{-\beta_k t/\sigma} \theta(\epsilon_k t), \tag{74}$$

where we have defined constants $B_1, \ldots, B_{N-1}$ and $\epsilon_1, \ldots, \epsilon_{N-1}$ by

$$B_k \equiv -(\epsilon_k/\eta c_{N-1}) \prod_{j \neq k} (\beta_k - \beta_j)^{-1}, \qquad \epsilon_k \equiv \begin{cases} +1 & \text{if} \quad \mathrm{Re}\,\beta_k > 0 \\ -1 & \text{if} \quad \mathrm{Re}\,\beta_k < 0 \end{cases} \tag{75}$$

Using the Green function, we can express the solution to the approximate equation of motion (66) as an integro-differential equation:

$$v(t) = (1/m) \int G(t - t')F_{\text{ext}}(t')\, \mathrm{d}t' \tag{76}$$

$$= (1/m) \sum_{k=1}^{N-1} B_k \int \theta(\epsilon_k \tau)\, \mathrm{e}^{-\beta_k \tau/\sigma} F_{\text{ext}}(t - \tau)\, \mathrm{d}\tau. \tag{77}$$

As for electrodynamics, by expressing the solution to a higher-order equation of motion in terms of a suitable Green function we have eliminated the runaway solutions; for example, equation (76) implies that $v(t) = 0$ for $F_{\text{ext}} = 0$. Note that terms with $\epsilon_k = 1$ only depend on the value of the external force at times $t' < t$, and thus describe retarded effects, while terms with $\epsilon_k = -1$ only depend on the value of the external force at times $t' > t$, and thus describe advanced effects. The advanced effects are not present for the exact equation of motion (56); rather, they are an artifact of the approximations used to obtain equation (76).

From equation (69), it follows that in the limit of small particle size there will be advanced effects if $N \geqslant 4$. Let us compare this result with the analogous result for electrodynamics. Recall that the low-velocity limit of the electrodynamic self-force is given by equation (63). It is the point particle limit of this self-force that appears in the equation of motion (18), which, as we showed in section 2, gives rise to advanced effects. But taking the point particle limit of equation (63) is equivalent to truncating a series expansion of the self-force at order $N = 3$. Thus, advanced effects show up at third order for electrodynamics, but only at fourth order for the toy model. This can be understood by comparing equations (62) and (63) for the self-forces in the two theories. For both theories, the sign of the third-order term is determined by the sign of τ_0: for electrodynamics τ_0 is positive, corresponding to advanced effects, while for the toy model τ_0 is negative, corresponding to retarded effects. Also, note that for the toy model the higher-order terms in the self-force vanish in the point particle limit, and thus the spurious advanced effects exhibited by equation (76) go away in this limit. For electrodynamics, however, the third-order term of the self-force is independent of the particle size, and hence the advanced effects remain in the point particle limit.
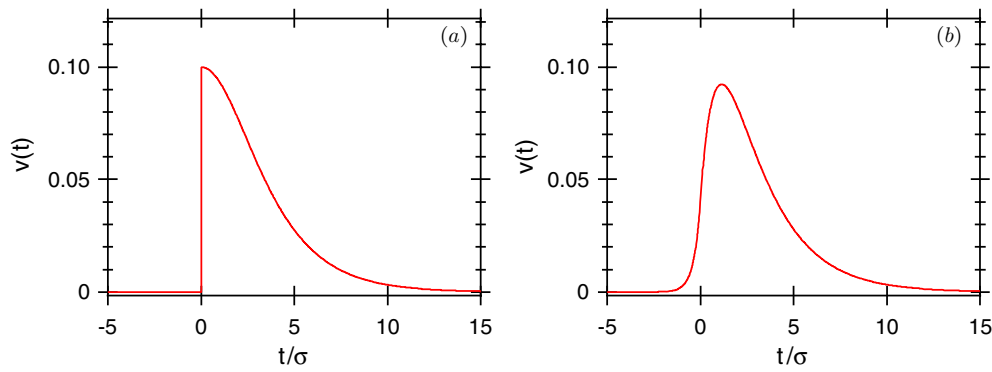
Figure 4. Graph of $v(t)$ versus t/σ for an impulsive force ($\eta = 0.25$, $v_0 = 0.1$). (*a*) Exact equation of motion (56). (*b*) Approximate integro-differential equation of motion (76) with $N = 4$.

As an example, let us consider the special case of an impulsive force $F_{\text{ext}}(t) = mv_0\delta(t)$, and compare the evolution predicted by the exact equation of motion (56) with the evolution predicted by the approximate integro-differential equation of motion (76). Suppose that the particle starts at rest at the origin, so $z(t) = 0$ for $t \leqslant 0$. The moment after the impulsive force is applied, the velocity of the particle is $v(0) = v_0$. We can numerically integrate the exact equation of motion (56) subject to these initial conditions; the result is shown in figure 4(*a*), where we have taken $\eta = 0.25$ and $v_0 = 0.1$. As expected, there are no advanced effects: the particle does not respond to the impulsive force until after it has been applied. We can also describe the evolution of the particle using the approximate integro-differential equation (76). If we substitute for $F_{\text{ext}}(t)$, we find that

$$v(t) = v_0 \sum_{k=1}^{N-1} B_k\, e^{-\beta_k t/\sigma} \theta(\epsilon_k t). \tag{78}$$

In figure 4(*b*) this solution is shown for $\eta = 0.25$, $v_0 = 0.1$, $N = 4$. Now there are advanced effects: the particle accelerates before the impulsive force has been applied.

6. Conclusion

We have shown that the advanced effects in classical electrodynamics are due to the fact that the planes of simultaneity for an extended particle can tilt in such a way that the self-force exerted on the particle at proper time τ depends on the state of the particle at proper times $\tau' > \tau$. We have also considered a toy model of electrodynamics in which the planes of simultaneity do not tilt and have shown that it does not give rise to advanced effects.

We have derived three equations of motion for electrodynamics. Equation (35) is an exact equation of motion for an extended particle; it is a second-order integro-differential equation that exhibits advanced effects. Equation (17), the Lorentz–Dirac equation, approximates equation (35) in the limit of a small particle; it is a third-order ordinary differential equation and admits runaway solutions. Equation (25) is obtained from the Lorentz–Dirac equation by choosing the initial conditions so as to eliminate the runaway solutions. It also approximates equation (35) in the limit of a small particle, and like equation (35) it is a second-order integro-differential equation that exhibits advanced effects.

We have derived four equations of motion for the toy model, which can be viewed as analogs to the various electrodynamic equations of motion. Equation (56) is an exact equation of motion for an extended particle; it is a second-order integro-differential equation, and does not exhibit advanced effects. Equation (51) is an exact equation of motion for a point particle; it is a second-order ordinary differential equation and does not exhibit advanced effects. Equation (66) approximates equation (56) in the limit of a small particle; it is an ordinary differential equation of order N, and admits runaway solutions for $N \geqslant 4$. Equation (76) is obtained from equation (66) by choosing the initial conditions so as to eliminate the runaway solutions; it is a first-order integro-differential equation, and exhibits spurious advanced effects for $N \geqslant 4$.

Appendix A. Self-energy of an extended particle

In this appendix we show that the self-force given in equation (32) can be used to derive the correct expression for the self-energy of a spatially extended particle. To accomplish this, we will use equation (32) to calculate the self-force that acts on a spatially extended particle undergoing uniformly accelerated motion. For simplicity we will work to first order in the acceleration and calculate the self-force at the moment at which the particle is instantaneously at rest. The trajectory, velocity and acceleration of the particle are

$$z^\mu(\tau) = (\tau, \mathbf{a}\tau^2/2), \qquad v^\mu(\tau) = (1, \mathbf{a}\tau), \qquad a^\mu(\tau) = (0, \mathbf{a}). \tag{A.1}$$

From equation (28), we find that

$$\bar{J}^\mu(x, \tau) = e(1 + \mathbf{a} \cdot \mathbf{x}) f(|\mathbf{x} - \mathbf{z}(\tau)|^2) v^\mu(\tau) \delta(t - (1 + \mathbf{a} \cdot \mathbf{x})\tau), \tag{A.2}$$

where $t \equiv x^0$. Substituting $\bar{J}^\mu(x, \tau)$ into equation (27), we find that the current density is given by

$$J^\mu(x) = e \int f(|\mathbf{x} - \mathbf{z}(\tau)|^2) v^\mu(\tau) \delta(\tau - (1 + \mathbf{a} \cdot \mathbf{x})^{-1} t) \, d\tau, \tag{A.3}$$

so $J^0(x) = \rho(t, \mathbf{x})$ and $\mathbf{J}(x) = \mathbf{a} t \rho(t, \mathbf{x})$, where we have defined

$$\rho(t, \mathbf{x}) \equiv e f(|\mathbf{x} - \mathbf{a} t^2/2|^2). \tag{A.4}$$

Note that

$$\bar{J}^\mu(x, 0) = (1 + \mathbf{a} \cdot \mathbf{x}) \rho(0, \mathbf{x}) \delta^\mu{}_0 \delta(t). \tag{A.5}$$

Substituting equation (A.5) into equation (32), we find that the self-force at $\tau = 0$ is

$$\mathbf{K}_r(0) = \int \rho(0, \mathbf{x}) \mathbf{E}_r(0, \mathbf{x}) \, d^3x + \int \mathbf{a} \cdot \mathbf{x} \rho(0, \mathbf{x}) \mathbf{E}_r(0, \mathbf{x}) \, d^3x, \tag{A.6}$$

where $\mathbf{E}_r(t, \mathbf{x})$, the retarded electric field, is given by

$$\mathbf{E}_r(t, \mathbf{x}) = -\nabla A_r^0(t, \mathbf{x}) - \partial_t \mathbf{A}_r(t, \mathbf{x}), \tag{A.7}$$

and $A_r^\mu(x)$, the retarded vector potential, can be obtained from the current density $J^\mu(x)$ via equation (6). After a lengthy but straightforward calculation, we find that

$$\mathbf{E}_r(t, \mathbf{x}) = -\nabla \int |\mathbf{x} - \mathbf{x}'|^{-1} \rho(t, \mathbf{x}') \, d^3x' - \int \left(\mathbf{a} |\mathbf{x} - \mathbf{x}'|^{-1} - \frac{1}{2} (\mathbf{a} \cdot \nabla) \nabla |\mathbf{x} - \mathbf{x}'| \right) \rho(t, \mathbf{x}') \, d^3x'. \tag{A.8}$$
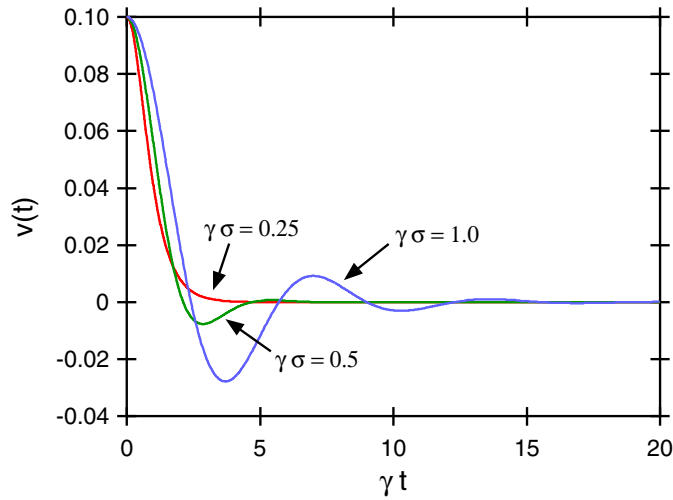
Figure B1. Graph of $v(t)$ versus γt for an impulsive force ($v_0 = 0.1$). Red curve: $\gamma \sigma = 0.25$. Green curve: $\gamma \sigma = 0.5$. Blue curve: $\gamma \sigma = 1.0$.

Substituting this result into equation (A.6), we find that the self-force is

$$\mathbf{K}_r(0) = -m_S \mathbf{a}, \tag{A.9}$$

where the self-energy m_S is given by

$$m_S = (e^2/2) \iint |\mathbf{x} - \mathbf{x}'|^{-1} f(|\mathbf{x}|^2) f(|\mathbf{x}'|^2) \, d^3x \, d^3x'. \tag{A.10}$$

It is interesting to note that the Abraham-Lorentz self-force corresponds to just the first term of equation (A.6), and gives an incorrect value of $(4/3)m_S$ for the self-energy (see [8], and section 17.3 of [2]). The correct relativistic expression for the self-force given in equation (32) modifies this result by producing the second term of equation (A.6), which evaluates to $-(1/3)m_S$ and combines with the first term to give the correct value for the self-energy.

Appendix B. Example solutions for a spatially extended particle

In this appendix we consider some example solutions to equation (56), the exact equation of motion for an extended particle in the toy model. Note that one region of an extended particle can cause a change in the field that acts back on a different region of the particle at a later time; as we shall see, if the particle is large enough the delay between these events can lead to oscillatory behavior.

As for the example in section 5.2, we will assume that the particle is initially at rest at the origin, and that it is driven with an impulsive force $F_{\text{ext}}(t) = mv_0\delta(t)$; this is equivalent to taking the initial conditions of the particle to be $v(0) = v_0$, $z(t) = 0$ for $t \leqslant 0$. We numerically integrate the equation of motion (56) subject to these initial conditions, and plot $v(t)$ versus γt in figure B1. Curves are shown for three different values of the particle size σ. Note that the renormalized mass of an extended particle is $m_R = m + m_S = (1 - 2\gamma\sigma/\sqrt{\pi})m$, so for the curve with $\gamma\sigma = 1$ the renormalized mass is negative.

Appendix C. Solution for $N = 3$

Here we find the solutions to equation (66), the approximate equation of motion for the toy model, for the case $N = 3$. We will assume there is no external driving force ($F_{ext} = 0$), so we can express the equation of motion as

$$\dot{v} = -\gamma_R v + \tau_R \ddot{v}, \tag{C.1}$$

where $\tau_R \equiv (m/m_R)\tau_0$, $\gamma_R \equiv (m/m_R)\gamma$, $m_R \equiv m + m_S$. The solutions are given by

$$v(t) = (\alpha_+ - \alpha_-)^{-1}((\alpha_+ v_0 + a_0)\,\mathrm{e}^{-\alpha_- t} - (\alpha_- v_0 + a_0)\,\mathrm{e}^{-\alpha_+ t}), \tag{C.2}$$

where $\alpha_\pm \equiv -(1/2\tau_R)(1 \pm (1 + 4\gamma_R \tau_R)^{1/2})$, and v_0 and a_0 are the initial velocity and acceleration.

References

[1] Rohrlich F 2007 *Classical charged particles,* 3rd edn (New Jersey: World Scientific)
[2] Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley)
[3] Barut A O 1980 *Electrodynamics and classical theory of fields and psarticles* (New York: Dover)
[4] Dirac P A M 1938 *Proc. R. Soc.* A **167** 148–69
[5] Rohrlich F 1961 *Ann. Phys.* **13** 93–109
[6] Boozer A D 2007 *Eur. J. Phys.* **28** 447–64
[7] Boozer A D 2007 *Eur. J. Phys.* **28** 1131–43
[8] Lorentz H A 1952 *Theory of Electrons* 2nd edn (New York: Dover)